11. V. M. Paskonov, "On an algorithm to solve boundary-layer problems," in: Computational Methods and Programming [in Russian], Vol. 7, Izd. Mosk. Gos. Univ. (1967).
12. V. V. Sapelkin, "Conjugate problem of unsteady heat transfer between laminar boundary layer and flat plate with internal heat source," Teplofiz. Vys. Temp., No. 6 (1981).
13. A. A. Samarskii and E. S. Nikolaev, Methods of Solving Finite-Difference Equations [in Russian], Nauka, Moscow (1978).
14. D. I. Didenko, "Computation of heat transfer between a flat plate heated at the ends and the surrounding fluid flow," Teplofiz. Teplotekh., 23 (1973).
15. M. Soliman and H. A. Johnson, "Transient heat transfer for turbulent flow over a flat plate of appreciable thermal capacity and containing time-dependent heat source," Trans. ASME, Ser. C, J. Heat Transfer, 89, No. 4 (1967).
16. R. C. C. Wang, B. T. F. Chung, and L. C. Thomas, "Transient convective heat transfer for laminar boundary layer flow with effects of wall capacitance and resistance," Trans. ASME, Ser. C, J. Heat Transfer, 99, No. 4 (1977).
17. R. G. Kemink and E. M. Sparrow, "Heat transfer coefficients for melting about a vertical cylinder with or without subcooling and for open or closed containment," Int. J. Heat Mass Transfer, 24, No. 10 (1981).
18. B. Sunden, "Conjugated heat transfer from circular cylinders in low Reynolds number flow," Int. J. Heat Mass Transfer, 23, No. 10 (1980).

FLOW STRUCTURE NEAR THE TRAILING EDGE OF A PLATE

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Solutions were obtained in [1-3] for the vicinity of the trailing edge of a plane plate at high but precritical Reynolds numbers Re_0, calculated with the plate length l and incident flow parameters, for subsonic and supersonic external flows, which describe the motion in a transition region of extent $x \sim O(l\,Re_0^{-3/8})$ between the known Blasius flow on the plane plate and the flow in the wake [4]. These solutions have a singularity in the wake behind the plate, which can be overcome by the use of numerical methods. The presence of this singularity indicates the need to study the flow in the region $x < l\,Re_0^{-3/8}$.

The present study will use the method of combined asymptotic expansions as $Re_0 \to \infty$ to study the flow near the trailing edge of a plate within the region $lRe_0^{-3/4} < x < lRe_0^{-3/8}$. It is found that at such lengths in the region near the plate a "compensation" flow regime is realized [5], wherein the solutions of [1-3] are valid for the rear edge of the plate, and a singularity of the former type exists in the wake. It is shown that in the singular region at $x \sim O(lRe_0^{-3/4})$, in a first approximation the flow may be described by the Navier–Stokes equations for an incompressible liquid. Numerical solutions are obtained for a thin plate and a thick plate over a wide range of local Reynolds number Re = 0-100. Flow line patterns, detachment zone characteristics, and gas dynamic flow function distributions over the surface of the bodies are presented.

1. In constructing the solutions of [1-3] to evaluate flow functions in the narrow region near the surface of the plate, it was assumed that the flow functions change in proportion to distance from the plate surface, that the flow was viscous, and that the discontinuity in boundary conditions at the trailing edge of the plate produced nonlinear perturbations of the flow functions. Then, using the equations of motion of the liquid, we easily obtain

$$u \sim x^{1/3},\ v \sim \varepsilon x^{-1/3},\ \Delta p \sim x^{2/3},\ \delta \sim \varepsilon x^{1/3}. \qquad (1.1)$$

Here and below we will use dimensionless variables; for this purpose all linear dimensions are referred to l, pressure and enthalpy to $\rho_0 u_0^2$ and u_0^2, respectively; the remaining flow functions are referred to their values in the unperturbed incident flow; δ is the thickness of the mixing layer behind the plate edge; $\varepsilon = Re_0^{-1/2}$.

In the flow under consideration the origin of mixing layer formation x = 0 is fixed, and so Eq. (1.1) describes a singularity immediately behind the trailing edge of the plate. Equation (1.1) is complemented by the conditions of interaction in the layer near the plate

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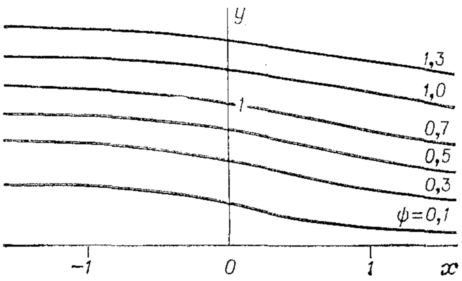

Fig. 1

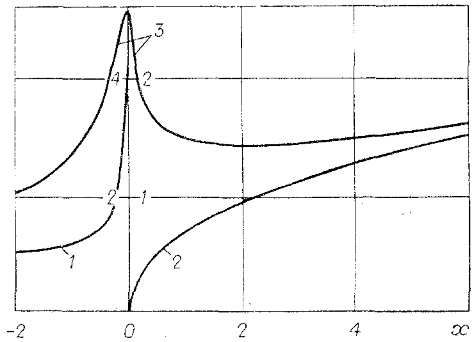

Fig. 2

with the external subsonic or supersonic flow $\Delta p \sim \delta/x$, and then we obtain estimates for scales and flow functions in the wall region (region III of [1-3]) $x \sim \varepsilon^{3/4}$, $y \sim \varepsilon^{5/4}$, $u \sim \varepsilon^{1/4}$, $v \sim \varepsilon^{3/4}$, $\Delta p \sim \varepsilon^{1/2}$.

If we now consider regions of length $\varepsilon^{3/2} < x < \varepsilon^{3/4}$, i.e., shorter than in [1-3], Eq. (1.1) remains in force, and the flow on the wall will interact with the wall portion of the boundary layer at the trailing edge of the plate [5]. We then have a "compensation" type flow, wherein the interaction conditions are localized in character, and the flow near the plate remains unperturbed, i.e., like that of [1-3] at the trailing edge of the plate, and the solution for the wake is in fact the first term of the coordinate expansion of the solution for a region $x \sim \varepsilon^{3/4}$ in extent (3).

2. The considerations of Sec. 1 permit construction of a solution of the Navier–Stokes equation in the vicinity of the singular region, where the longitudinal and transverse velocity components become equal in order of magnitude. It follows from Eq. (1.1) that this is valid in a region with characteristic dimensions $x \sim y \sim O(\varepsilon^{3/2})$, for which it is necessary to introduce new independent variables and asymptotic expansions for the flow functions:

$$
\begin{gathered}
x = \varepsilon^{3/2}x_1,\ y = \varepsilon^{3/2}y_1,\\
u(x,\ y;\ \varepsilon) = \varepsilon^{1/2}u_1(x_1,\ y_1) + \ldots,\ v(x,\ y;\ \varepsilon) = \varepsilon^{1/2}v_1(x_1,\ y_1) + \ldots,\\
p\,(x, y; \varepsilon) = 1/\gamma\,\mathrm{M}_0^2 + \varepsilon^{1/2}p_w + \varepsilon p_1(x_1, y_1) + \ldots,\ \mu\,(x, y;\ \varepsilon) = \mu_w + \ldots,\\
\rho(x,\ y;\ \varepsilon) = \rho_w + \ldots,\ h(x,\ y;\ \varepsilon) = h_w + \varepsilon^{1/2}h_1(x_1,\ y_1) + \ldots
\end{gathered} \tag{2.1}
$$

Here all variables are conventional; the subscript w denotes variables on the plate surface at the trailing edge, which will differ for subsonic and supersonic external incident flows.

Substitution of the expansions of Eq. (2.1) in the Navier–Stokes equations and performance of the limiting transform as $\varepsilon \to 0$ shows that in the first approximation the flow in the region of the trailing edge of the plate with characteristic dimensions $x \sim y \sim O(\varepsilon^{3/2})$ is described by Navier–Stokes equations for an imcompressible liquid:

$$
\begin{gathered}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),\quad u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = \frac{1}{\mathrm{RePr}}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right),\\
\mathrm{Re} = \rho_w A a_1^2/\mu_w,\quad A = (\partial u/\partial y)_w,\quad B = (\partial h/\partial y)_w.
\end{gathered} \tag{2.2}
$$

Here the coordinates x, y are refered to some dimension in the flow region a_1; the velocity components u, v, and the perturbations in enthalpy h and pressure p are refered to their values and twice the velocity head in the incident shear flow at a distance a_1 from the plate surface, respectively; Re is the local Reynolds number; Pr, the Prandtl number, is taken equal to 0.7 in all calculations. In Eq. (2.2) and below, we will omit the subscript 1 for simplicity.

On the surface of the body flowed over the normal nonpenetration and adhesion conditions must be satisfied:

$$
u = v = 0, \tag{2.3}
$$

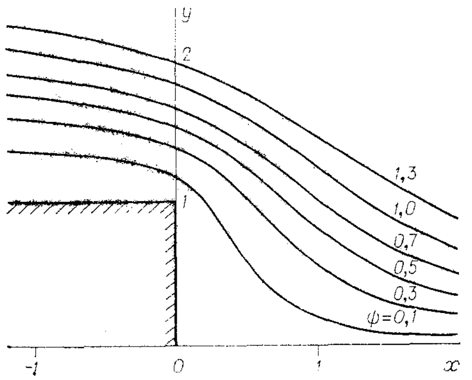

Fig. 3

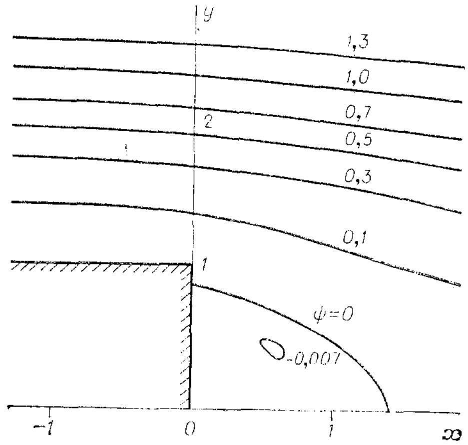

Fig. 4

while on the symmetry line in the wake the conditions of symmetry and smoothness of the profiles of the functions u, v, and h must be satisfied:

$$\partial u/\partial y = v = \partial h/\partial y = 0. \tag{2.4}$$

The external boundary conditions are obtained by union with the incident shear flow:

$$u \to y,\ h \to y\ (x \to -\infty \quad \text{or} \quad y \to \infty), \tag{2.5}$$

in the wake the solution for the region under consideration must transform to an expression of the form of Eq. (1.1):

$$u \sim x^{1/3},\ h \sim x^{1/3} (x \to \infty). \tag{2.6}$$

In the notation used, the dimensionless shear stress τ and thermal flux q are expressed by

$$\tau = \tau_{xy}/\varepsilon\rho_0 u_0^2 \mu_w A = \partial u/\partial y + \partial v/\partial x,\ q = -q_w \,\mathrm{Pr}/\varepsilon\rho_0 u_0^3 \mu_w B = \partial h/\partial n$$

(where n is the external normal to the body surface) and in the incident shear flow $\tau = q = 1$.

The boundary problem of Eqs. (2.2)-(2.6) without the equation of conservation of energy was formulated and partially studied in [6].

3. The boundary problem of Eqs. (2.2)-(2.6) was solved numerically in traditional flow function and vorticity variables. The method for solution of such problems was detailed in [7].

In flow over a plane pate there is no characteristic length in the boundary problem, and we may take $a_1 = (\mu_w/A\rho_w)^{1/2}$, with the local Reynolds number Re = 1. In this case the flow is undetached everywhere, and flow acceleration in the wake behind the plate produces a significant shift in flow lines toward the line of symmetry (Fig. 1). The shear stress τ increases abruptly upon approach to the trailing edge of the plate and agrees qualitatively with the expression presented in [8]: $\tau \sim x^{-2}$ as $x \to 0$ (curve 1, Fig. 2, left-hand ordinate scale). The longitudinal velocity u on the symmetry line (curve 2, Fig. 2, right-hand ordinate scale) at $x \leqslant 0.5$ agrees well with the expression of [8]: $u \sim x^{1/2}$ as $x \to 0$, while at $x \geqslant 10$ it agrees with the asymptotic expression (1.1).

In the motion studied the flow accelerates due to the action of viscosity forces in the mixing layer behind the plate. Near the plate surface the velocities are low and the flow here is described by the Stokes equation, i.e., viscous forces must be compensated by pressure forces. Therefore, the increase in the value of τ with approach to the trailing edge of the plate is accompanied by an increase in perturbation of the pressure p (curve 3, Fig. 2, left-hand ordinate scale). However, directly beyond the trailing edge, where the velocities are still low, in view of boundary conditions (2.4) the viscosity forces decrease rapidly, which leads to a corresponding increase in pressure at $x \geqslant 0$. At some distance from the trailing edge the flow in the mixing layer will still be described by the boundary-layer equations for a "compensated" flow regime [5]. Then flow acceleration in the mixing region and shifting of flow lines toward the line of symmetry cause braking of the external subsonic portion of the boundary layer and a corresponding increase in pressure, which at $x \geqslant 10$ agrees well with the asymptotic Eq. (1.1).

The calculations performed permit determination of the change in resistance of one side of a plane plate due to change in the shear stress τ in a singular region of extent $x \sim O(\varepsilon^{3/2})$ which is characterized by the value of $\tau_1 = \int_{-\infty}^{0} (\tau - 1)\,dx \approx 1.031.$

The distribution of the thermal flux q over plate surface differs little from the distribution of shear stress τ, and is not shown in Fig. 2. The quantity $q_1 = \int_{-\infty}^{0} (q - 1)\,dx \approx 1.066$ characterizes the change in heating of one side of the plate in the vicinity of the trailing edge. The perturbation in enthalpy h in the wake behind the plate on the symmetry line changes in practically the same manner as u, and so is also not shown in Fig. 2.

Boundary problem (2.2)-(2.6) also describes flow over the trailing edge of a plate with characteristic thickness $a \sim O(\varepsilon^{3/2})$, since the solution for the "compensated" flow regime in a region with characteristic dimensions $\varepsilon^{3/2} < x < \varepsilon^{3/4}$, $y \sim \varepsilon x^{1/3}$ in the first approximation as $\varepsilon \to 0$ remains unchanged. In this case, for the characteristic length we choose one half the plate thickness ($a_1 = \varepsilon^{3/2} a/2$, $a_1 \sim O(1)$). In performing the calculations the previous numerical scheme was used, with local Reyolds number varied over a wide range (Re = 0-100).

Figures 3, 4 show the flow line distribution in the flow field at Re = 0 and 3. The solution at Re = 0 corresponds to the Stokes limit, the flow being undetached in this case. At Re > 0 a detachment zone is formed, the length of which L increases practically in proportion to the local Reynolds number value: L ≈ 0.42 Re (see Table 1). The transverse dimension of the detachment zone changes insignificantly and at practically all Re the detachment zone begins barely below the upper edge of the plate section (for example, y = 0.88 for Re = 3 and y = 0.96 at Re = 100).

TABLE 1

Re	3	10	30	40	50
L	1,4	4,2	12,2	16,1	20
τ_1	0,799	0,557	0,451	0,408	0,355
q_1	0,465	0,287	0,207	0,180	0,165
q_2	0,968	0,781	0,655	0,650	0,650
p_2	−0,447	0,102	0,386	0,442	0,472

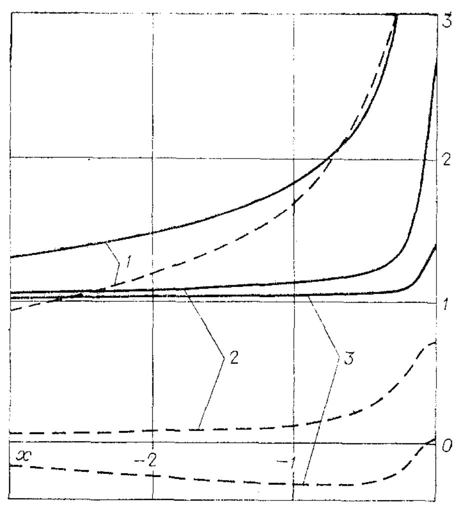

Fig. 5

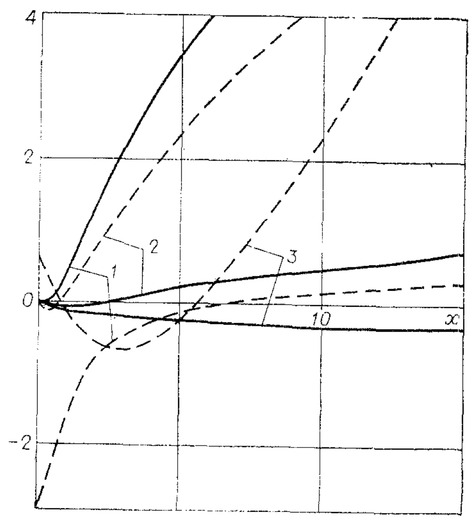

Fig. 6

The distribution of shear stress τ (solid lines) and pressure perturbation Rep (dashed lines) over plate surface for local Reynolds numbers Re = 0, 3, and 100 (curves 1-3) are shown in Fig. 5. It is evident that with growth in Re the perturbations in τ and p decrease, since due to increase in the extent of the detachment zone, the rear profile flowed over becomes ever smoother. This same fact explains the localization of flow perturbation near the edge of the plate section with increase in Re.

Figure 6 shows the change in longitudinal velocity u (solid lines) and pressure perturbation Rep (dashed lines) along the line of symmetry for the same Re values as in Fig. 5. It is clearly evident that with increase in Re the extent of the reverse flow region increases. However, the limited size of the computation region does not permit tracing flow function behavior all the way to their exit to asymptotic values, Eq. (1.1).

Table 1 also shows values of the quantities $\tau_1, q_1, q_2 = \int\limits_0^1 q\,dy$ (which characterizes heating of the plate face) and $p_2 = \mathrm{Re}\int\limits_0^1 p\,dy$ (which characterizes the pressure resistance of the plate face) for various values of Re. It follows from these results that the pressure resistance of a thick plate $p_2 = 0$ at Re $\approx$ 7.5.

LITERATURE CITED

1. K. Stewartson, "On the flow near the trailing edge of a flat plate. II," Matematika, 16, (1969).
2. A. F. Messiter, "Boundary layer flow near the trailing edge of a flat plate," SIAM J. Appl. Math., 18 (1970).
3. P. G. Daniels, "Numerical and asymptotic solutions for the supersonic flow near the trailing edge of a flat plate," Q. J. Mech. Appl. Math., 27, Pt. 2, No. 5 (1974).
4. S. Goldstein, "Concerning some solutions of the boundary layer equations in hydrodynamics," Proc. Cambridge Phil. Soc., 26, No. 1 (1930).
5. V. V. Bogolepov and V. Ya. Neiland, "Study of local perturbations of viscous supersonic flows," in: Aeromechanics [in Russian], Nauka, Moscow (1976).
6. K. Stewartson, "On the flow near the trailing edge of a flat plate," Proc. Roy. Soc., A306 (1968).
7. V. V. Bogolepov, "Calculation of flow over a small step inclined toward the flow," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1983).
8. K. Stewartson, "Multistructured boundary layers on flat plates and related bodies," Adv. Appl. Mech., 14 (1974).